# *Pluralism & Proofs*

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THE UNIVERSITY OF MELBOURNE

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Motivation

Intuitionist, Dual-Intuitionist
and Classical Negation

Three Negations in One Logic …

… or One Negation in Three Logics

*motivation*

# *Logical Pluralism*

# MOTIVATION #1

# *Disjunctive Syllogism & Inconsistency*

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$$p \vee q, \neg p \vdash^{?} q$$

# *Disjunctive Syllogism & Inconsistency*

$$p \lor q, \neg p \vdash^{?} q$$

*Yes*, and *no*.

# MOTIVATION #2

# *Constructive Mathematics*

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$$\neg\neg p \vdash^{?} p$$

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*Yes*, and *no*.

# A Picture

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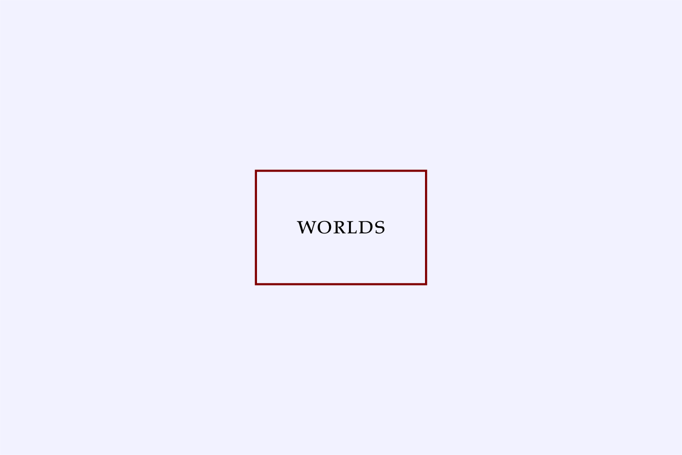

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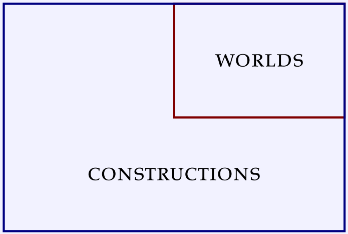

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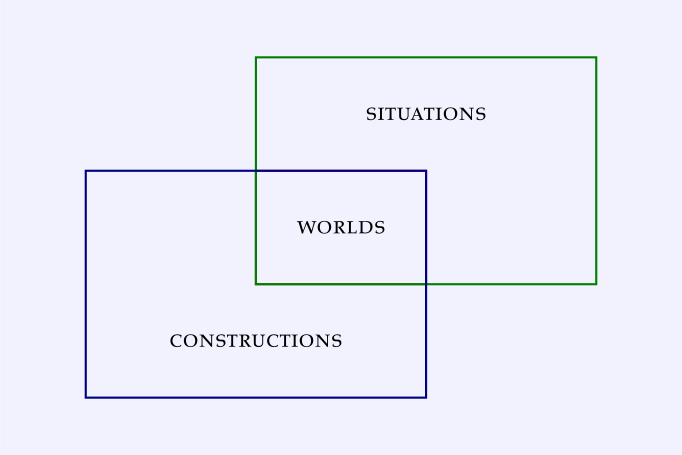

*One language,*
*different*
*consequence relations.*

# Logical Pluralism

JC Beall and Greg Restall

# REGRET #1

*Not much of an account of proof*

# regret #2

*Not much logic*

# REALISATION #1

*Lots of people are sympathetic with logical pluralism …*

# REALISATION #2

*. . . but not so many are pluralists in* ***our*** *sense*

*Articulate and explain the differences in pluralisms*

*and deal with the regrets*

*in one go.*

# My Mission

We'll use *proof* to explain the difference between

One Language, Many Logics

Many Languages, One Logic.

# *intuitionist dual-intuitionist & classical negation*

# Classical Negation

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,[\neg L] \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\,[\neg R]$$

# Intuitionist Negation

$$\frac{X \vdash A}{X, \neg A \vdash} [\neg L] \qquad \frac{X, A \vdash}{X \vdash \neg A} [\neg R]$$

# Dual-Intuitionist Negation

$$\frac{\vdash A, Y}{\neg A \vdash Y}\,[\neg L] \qquad \frac{A \vdash Y}{\vdash \neg A, Y}\,[\neg R]$$

# Classical Derivations

$$\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\,[\neg L]}{p \vdash \neg\neg p}\,[\neg R] \qquad \dfrac{\dfrac{p \vdash p}{\vdash \neg p, p}\,[\neg R]}{\neg\neg p \vdash p}\,[\neg L]$$

$$\dfrac{\dfrac{p \vdash p}{\vdash p, \neg p}\,[\neg R]}{\vdash p \vee \neg p}\,[\vee R] \qquad \dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\,[\neg L]}{p \wedge \neg p \vdash}\,[\wedge L]$$

# Intuitionist Derivations

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# Dual Intuitionist Derivations

$$\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\ [\neg L]}{p \vdash \neg\neg p}\ [\neg R]$$

$$\dfrac{\dfrac{p \vdash p}{\vdash \neg p, p}\ [\neg R]}{\neg\neg p \vdash p}\ [\neg L]$$

$$\dfrac{\dfrac{p \vdash p}{\vdash p, \neg p}\ [\neg R]}{\vdash p \lor \neg p}\ [\lor R]$$

$$\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\ [\neg L]}{p \land \neg p \vdash}\ [\land L]$$

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- A dual-intuitionist derivation *is* a classical derivation in which the negation steps satisfy the constraint: *at most one formula present on the* LHS.

# Virtues of Derivations

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- It can also have the virtue of being intuitionistically valid.
- It can also have the virtue of being dual-intuitionistically valid.
- And some have *all three* virtues.

# Another Picture

**CLASSICAL DERIVATIONS**

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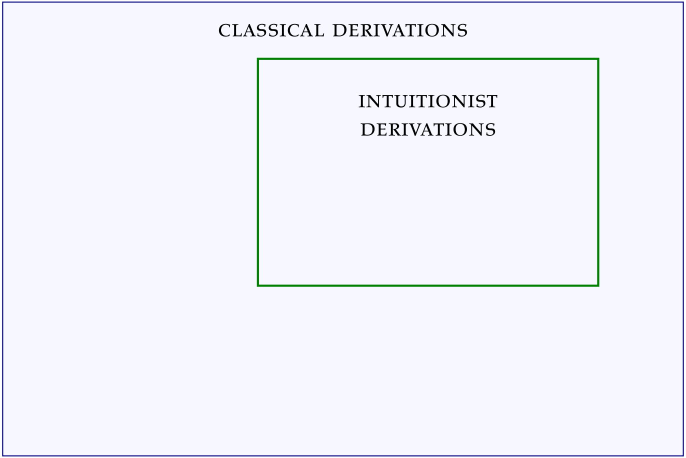

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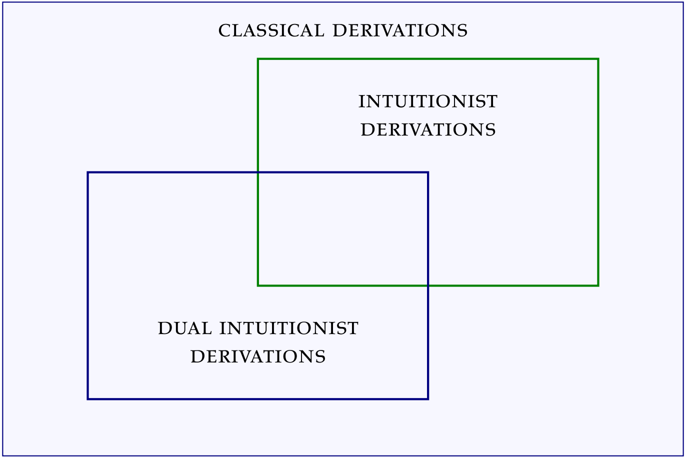

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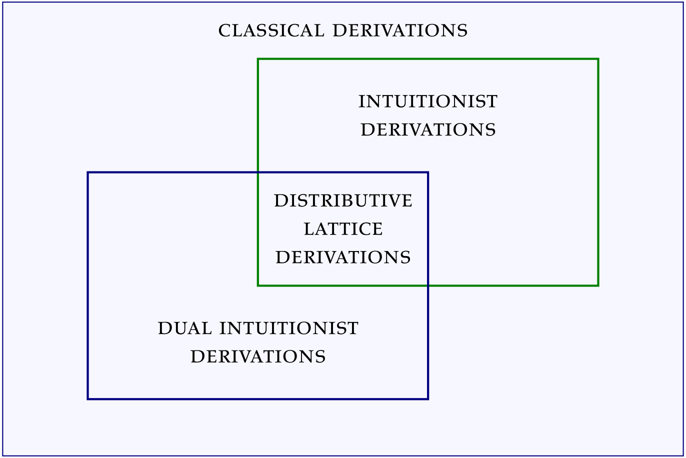

*One* language, *three* logics.

*three negations in one logic . . .*

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Now if $x \Vdash A$ and $x \leqslant x'$, then $x' \Vdash A$ too.

($X \vdash_{DJ} Y$ iff there's no $x$ where $x \Vdash A$ for each $A \in X$ and $x \nVdash B$ for each $B \in Y$)

# Hang on …

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and a point $x$ in that frame
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And these are all (very) different.

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- Does this make sense, from a pluralist perspective?
- Yes, for a Carnapian (you can be *bilingual*).
- Not so much, for me.

  At least, it's not the *only* way to understand the relationship between intuitionist, dual-intuitionist and classical negation.

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We'll look at them *proof theoretically* to give an example of a stance from which the three-negations-in-one-model is not so natural.

*. . . or one negation and three logics*

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This works for $\nvdash_K$ $\nvdash_J$ and $\nvdash_{DJ}$.

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- Dual Intuitionisically invalid? — A *???*

# Truth at Points

$A$ is true at $[\mathcal{X}, \mathcal{Y}]$ iff $A \in \mathcal{X}$.

# Conjunction

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Suppose $A$ and $B$ are in $\mathcal{X}$. Then since $A, B \vdash A \wedge B$, we cannot have $A \wedge B \in \mathcal{Y}$.

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Suppose $A$ and $B$ are in $\mathcal{X}$. Then since $A, B \vdash A \wedge B$, we cannot have $A \wedge B \in \mathcal{Y}$. So, since points are partitions, $A \wedge B$ is in $\mathcal{X}$.

# Disjunction

Exercise left to the listener.

HINT: dualise.

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Suppose $A$ is in $\mathcal{Y}$. Since $\vdash A, \neg A$, we cannot have $\neg A$ in $\mathcal{Y}$. So, since points are partitions, $\neg A$ is in $\mathcal{X}$.

This relies on having $A, \neg A \vdash$ and $\vdash A, \neg A$, so it works for $\vdash_K$, but neither $\vdash_J$ nor $\vdash_{DJ}$.

# For Warrants

FACT: If $[\mathcal{X}, \mathcal{Y}]$ is a *warrant*, then $\neg A$ is true at $[\mathcal{X}, \mathcal{Y}]$ iff $A$ is not true at any warrant $[\mathcal{X}', \mathcal{Y}']$ where $\mathcal{X} \subseteq \mathcal{X}'$.

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Suppose $\neg A$ is not in $\mathcal{X}$. So, since points are partitons, $\neg A$ in $\mathcal{Y}$. Then, since $X \not\vdash \neg A$ for any $X \subseteq \mathcal{X}$, we must have $X, A \not\vdash$ for any $X \subseteq \mathcal{X}$ too.

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Suppose $\neg A$ is not in $\mathcal{X}$. So, since points are partitons, $\neg A$ in $\mathcal{Y}$. Then, since $X \not\vdash \neg A$ for any $X \subseteq \mathcal{X}$, we must have $X, A \not\vdash$ for any $X \subseteq \mathcal{X}$ too. So, it follows that there is a point extending $[\mathcal{X} \cup \{A\}, \ ]$. This is a point $[\mathcal{X}', \mathcal{Y}']$ at which $A$ is true.

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# For Dual-Intuitionistically Invalid Points

FACT: If $[\mathcal{X}, \mathcal{Y}]$ is *dual-intuitionisically invalid,* then $\neg A$ is true at $[\mathcal{X}, \mathcal{Y}]$ iff $A$ is not true at some point $[\mathcal{X}', \mathcal{Y}']$ where $\mathcal{Y} \subseteq \mathcal{Y}'$.

# For Dual-Intuitionistically Invalid Points

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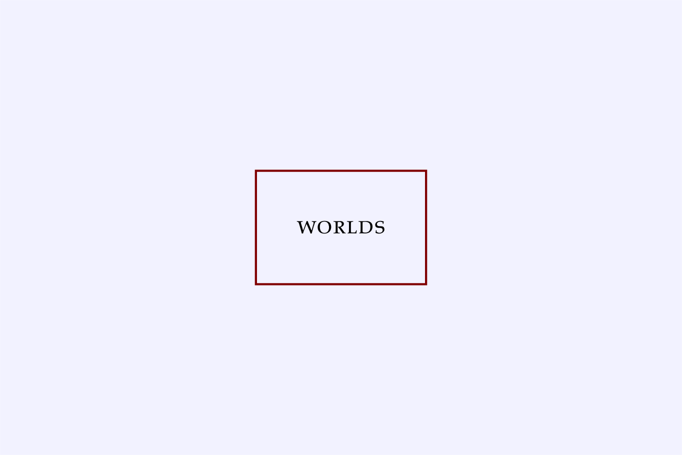
Proof: *dualise*.

# The Final Picture


WORLDS

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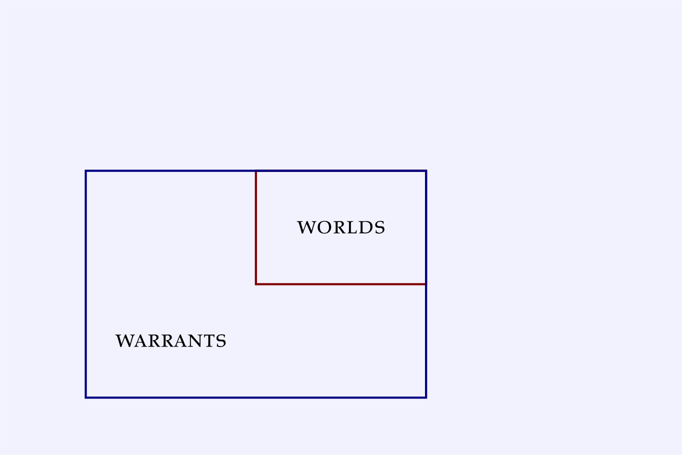

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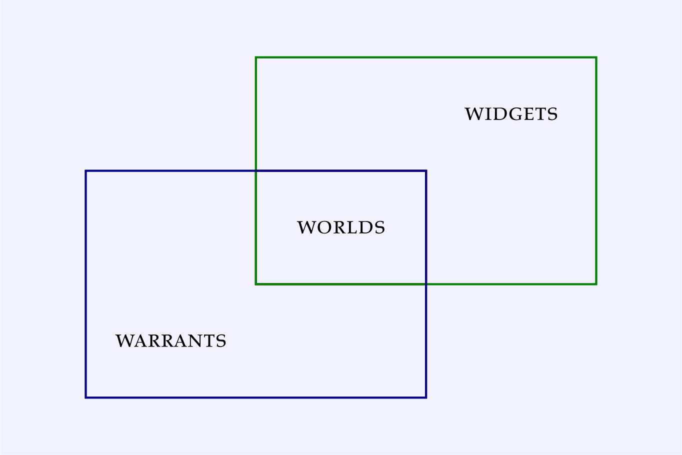

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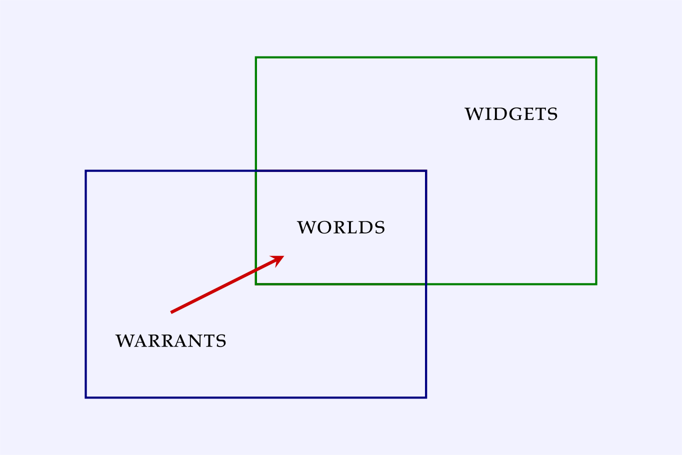

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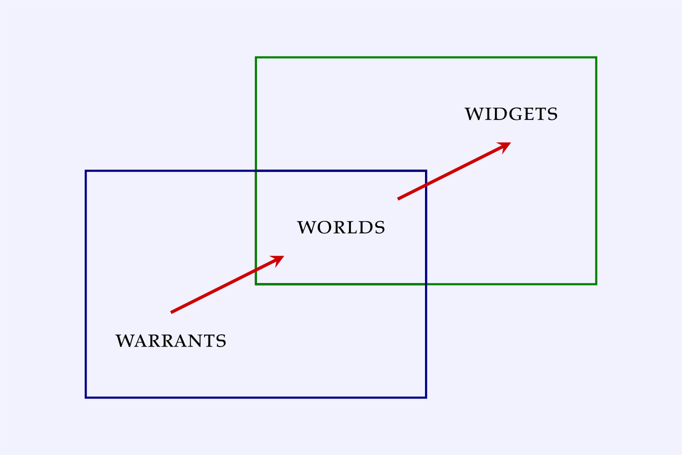

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*thank you!*